学位申請論文

大山陽介

Self-duality and Integrable Systems

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§0. Introduction

In his lectures (1984-85) at Kyoto University, Professor M.Sato presented a program for generalizing the soliton theory ([9]; c.f. [10]). The Kadomtsev-Petviashvili (KP) equation is a typical example of the soliton theory. The KP equation is written in the form of deformation equations of a linear ordinary differential equation. The time evolutions of a solution are interpreted as dynamical motions on an infinite dimensional Grassmann manifold ([7],[9]). The Lie algebra of microdifferential operators of one variable acts on this manifold transitively. He conjectured that any integrable systems can be written in the form of deformation equations of a linear systems, and proposed to investigate a deformation of differential equations in higher dimensions. He showed a simple example of a deformation of holonomic systems in higher dimensions ([9]), and its generalization is treated in [4]. In this paper we study a deformation of $\mathcal{D}$-modules in higher dimensions.

First we review the KP equation. We denote by $\mathcal{E}$ the ring of microdifferential operators of one variable x. We fix a microdifferential operator P, and denote by t_P a time variable with respect to P. We study the following evolution equation associated to P:

$$\frac{\partial W}{\partial t_P} + WP = (WPW^{-1})_+ W, \tag{0.1}$$

where $W = W(x, D_x) = 1 + \sum_{j<0} w_j(x) D_x^j \in \mathcal{E}$. We denote by $\mathcal{W}$ the set of such operators W. This space $\mathcal{W}$ is a group by the composition of $\mathcal{E}$. We get the KP-hierarchy taking $P = D_x^n \quad (n = 1, 2, 3, ...)$ in (0.1). The equation (0.1) defines a dynamical motion on $\mathcal{W}$. This infinitesimal action of the Lie algebra $\mathcal{E}$ on $\mathcal{W}$ is transitive.

The purpose of this article is to give a foundation for higher dimensional generalization of the KP hierarchy. Let now $\mathcal{E}$ be the ring of microdifferential operators in several variables. Similarly to the one dimensional case, fixing an operator $P \in \mathcal{E}$, we shall study the following equation

$$\frac{\partial W}{\partial t_P} + WP = (WPW^{-1})_+ W, \tag{0.2}$$

where the operator W is a 0-th order microdifferential operator. Here we choose a decomposition $\mathcal{E}= \mathcal{D} \oplus \mathcal{E}_\phi$ and $(WPW^{-1})_+ \in \mathcal{D}$ is the component of WPW^{-1} according to this decomposition. In general the equation (0.2) imposes some constraints on the initial value $W(t_P = 0)$, since the vector field defined by (0.2) is not tangent to the space $\mathcal{E}(0)$. There is no operator W_0 such that the equation (0.2) has a solution $W(t) \in \mathcal{E}$ with the initial value W_0 for any $P \in \mathcal{E}$. We take generators P in (0.2) only in the Lie subalgebra V of $\mathcal{E}$

$$
\begin{aligned}
V = \{F_0(x', D_0)x_0 + \sum_{0 \leq j < r} F_j(x', D_0)D_j + E(x', D_0);& \\
F_k(x', D_0),\ E(x', D_0) \in \mathcal{E} \quad \text{for} \quad 0 \leq k < r\},&
\end{aligned}
$$

where $x' = (x_1, x_2, ..., x_{r-1})$. This Lie algebra contains the transformation groups both of the self-dual Yang-Mills equations and of the self-dual Einstein equations (see [7],[8]). In §2 we will determine the subspace $\mathcal{W}$ of $\mathcal{E}(0)$ so that the vector field defined by (0.2) for any $P \in V$ is tangent to $\mathcal{W}$. The space $\mathcal{W}$ is a subgroup in $\mathcal{E}$. The Lie algebra V acts on $\mathcal{W}$ transitively.

In the case of $r = 3$, our integrable system is nothing but a composed system of the self-dual Yang-Mills equations and the equations of self-dual metrics on Riemannian manifolds of dimension four. The Lie algebra V acts transitively on the space of self-dual connections on self-dual spaces. Thus we obtain a group-theoretical description of the twistor theory ([1],[5]).

Notations. We use the following notations: $\mathbb{Z}$ denotes the set of integers. $\mathbb{N}$ denotes the set of non-negative integers. We denote by $\mathbb{C}$ the complex number field. We denote by 1_n the unit matrix of size $n \times n$.

§1. Deformations of $\mathcal{D}$-modules

Throughout this paper we shall work in the category of formal power series, $\mathcal{O} = \mathbb{C}[[x]] = \mathbb{C}[[x_0, x_1, \cdots, x_{r-1}]]$ $(r \geq 2)$. Let $\mathcal{D}$ be the ring of differential operators with coefficients in $\mathcal{O}$. Then every differential operator P of order m can be written as:

$$P = \sum_{\alpha \in \mathbb{N}^r, |\alpha| \leq m} a_\alpha(x) D_x^\alpha,$$

where $a_\alpha(x)$ are elements of $\mathcal{O}$, $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{r-1}) \in \mathbb{N}^r, |\alpha| = \alpha_0 + \alpha_1 + \cdot \cdot + \alpha_{r-1}$, $D_x^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \cdots D_{r-1}^{\alpha_{r-1}}$ and $D_j = \frac{\partial}{\partial x_j}$ $(j = 0, 1, \cdot \cdot, r-1)$.

The ring $\mathcal{E}$ of formal microdifferential operators is a set of formal Laurent series in $D_0, D_1, \cdots, D_{r-1}$ with only non-negative powers of $D_1, \cdots, D_{r-1}$. The precise definition is as follows. We denote by $\mathcal{E}(m)$ the space of formal series :

$$P = \sum_{\alpha \in \mathbb{Z} \times \mathbb{N}^{r-1}, |\alpha| \leq m} a_\alpha(x) D_x^\alpha$$

where a_α's are elements of $\mathcal{O}$, and the summation is taken through $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{r-1})$, where $\alpha_0 \in \mathbb{Z}, \alpha_1 \in \mathbb{N}, \cdots, \alpha_{r-1} \in \mathbb{N}$. We set

$$\mathcal{E} = \bigcup_{m \in \mathbb{Z}} \mathcal{E}(m).$$

We endow the $\mathcal{O}$-module $\mathcal{E}$ with a structure of ring by extending the Leibniz formula. For two elements $P = \sum_\alpha a_\alpha D_x^\alpha$ and $Q = \sum_\beta b_\beta D_x^\beta$ of $\mathcal{E}$, we define the composition $P \circ Q$ by

$$P \circ Q = \sum_{\alpha, \beta \in \mathbb{Z} \times \mathbb{N}^{r-1}, \gamma \in \mathbb{N}^r} \binom{\alpha}{\gamma} a_\alpha b_\beta^{(\gamma)} D_x^{\alpha+\beta-\gamma},$$

where $b_\beta^{(\gamma)} = D_x^\gamma(b_\beta)$. The ring $\mathcal{E}$ has an increasing filtration by subspaces $\{\mathcal{E}(m)\}_{m \in \mathbb{Z}}$. We have

$$\mathcal{E}(m)\mathcal{E}(n) = \mathcal{E}(n+m).$$

For any $\mathcal{O}$-submodule $\mathcal{L}$ of $\mathcal{E}$ we define the induced filtration $\{\mathcal{L}(m)\}_{m\in\mathbb{Z}}$ of $\mathcal{L}$ by $\mathcal{L}(m) = \mathcal{L} \cap \mathcal{E}(m)$.

Let $\mathcal{E}_\phi$ be the $\mathcal{O}$-module consisting of the formal microdifferential operators of the form

$$\sum_{\alpha_0<0} a_\alpha(x) D_x^\alpha,$$

The ring $\mathcal{E}$ is the direct sum of $\mathcal{D}$ and $\mathcal{E}_\phi$. For any $P \in \mathcal{E}$, we define $P_+ \in \mathcal{D}$ and $P_- \in \mathcal{E}_\phi$ by the decomposition of $\mathcal{E}$:

$$\begin{array}{ccccc} \mathcal{E} & = & \mathcal{D} & \oplus & \mathcal{E}_\phi \\ P & = & P_+ & + & P_- \end{array} .$$

For any $\mathcal{O}$-submodule $\mathcal{L}$ of $\mathcal{E}$ we define the $\mathcal{O}$-module $\mathcal{L}_-$ by $\mathcal{L}_- = \mathcal{L} \cap \mathcal{E}_\phi$. Remark that $\mathcal{E}(0) = \mathcal{O} \oplus \mathcal{E}_\phi(0)$.

In the following we shall study a left $\mathcal{D}$-submodule I of $\mathcal{E}$ which satisfies the following condition:

$$\mathcal{E}(m) = I(m) \oplus \mathcal{E}_\phi(m) \quad \text{for any} \quad m \in \mathbb{Z}. \tag{1.1}$$

For example $I = \mathcal{D}$ satisfies (1.1). We make clear the structure of such a $\mathcal{D}$-submodule I.

Lemma 1.1 Suppose that a $\mathcal{D}$-submodule I of $\mathcal{E}$ satisfies the condition (1.1). Then I is generated as $\mathcal{D}$-module by a unique operator W such that

$$W \in \mathcal{E}(0) \quad \text{and} \quad W_+ = 1. \tag{1.2}$$

Proof. The operator W is obtained by decomposing the identity operator $1 \in \mathcal{E}(0)$ into the sum of an operator in $I(0)$ and an operator in $\mathcal{E}_\phi(0)$ according to the condition (1.1):

$$\begin{array}{ccccc} \mathcal{E}(0) & = & I(0) & \oplus & \mathcal{E}_\phi(0) \\ 1 & = & W & + & U \end{array}$$

It is evident that W is contained in $\mathcal{E}(0)$ and that $W_+ = 1$. Since we have

$$\mathcal{E}(m)W = \mathcal{E}(m), \quad \mathcal{E}_\phi(m)W = \mathcal{E}_\phi(m)$$

for any $m \in \mathbb{Z}$, we obtain that

$$\mathcal{E}(m) = \mathcal{D}(m)W \oplus \mathcal{E}_\phi(m) \quad \text{for any } m \in \mathbb{Z}.$$

Thus the $\mathcal{D}$-module $I' = \mathcal{D}W$ also satisfies (1.1). Because I contains I' and both satisfy (1.1), I coincides with I'. The uniqueness is clear. $\square$

Remark that W in Lemma 1.1 is invertible by (1.2).

We investigate nonlinear evolution equations according to the program of M. Sato ([9],[10]). For any $P \in \mathcal{E}$ and any $\mathcal{D}$-submodule I_0 of $\mathcal{E}$ we define the time evolution I_t of I_0 by the following differential equations:

$$\frac{\partial V(t)}{\partial t} + V(t)P \in I_t \qquad \text{for any} \quad V(t) \in I_t. \tag{1.3}$$

We call $P \in \mathcal{E}$ the generator of the evolution equation (1.3).

In general we cannot find any $\mathcal{D}$-submodule I_t which solves (1.3). In this paper we shall study the case that we can find a solution I_t of (1.3) which is a $\mathcal{D}$-module satisfying (1.1) for any t. Then I_t is generated by an operator $W(t) \in \mathcal{E}(0)$ by Lemma 1.1 and we can rewrite the equation (1.3) in terms of the generator $W(t)$.

Lemma 1.2 We fix an operator $P \in \mathcal{E}$. We assume that the solution I_t of the evolution equation (1.3) is a $\mathcal{D}$-submodule which satisfies (1.1) for any t . Then the equation (1.3) reduces to the following equation

$$\frac{\partial W(t)}{\partial t} + W(t)P = (W(t)PW(t)^{-1})_+ W(t), \tag{1.4}$$

where the operator $W(t)$ is the generator of I_t in Lemma 1.1.

Proof. From the equation (1.3) there exists an operator $B(t) \in \mathcal{D}$ such that

$$\frac{\partial W(t)}{\partial t} + W(t)P = B(t)W(t).$$

Thus we have

$$B(t) = W(t)PW(t)^{-1} + \frac{\partial W(t)}{\partial t} W(t)^{-1}.$$

Since the operator $W(t)$ is contained in $1 + \mathcal{E}_\phi$, the operator $\frac{\partial W(t)}{\partial t} W(t)^{-1}$ is contained in $\mathcal{E}_\phi$. Thus we obtain that $B(t) = (W(t)PW(t)^{-1})_+$. □

Remark. The equation (1.4) is rewritten as

$$\begin{aligned} \frac{\partial W}{\partial t} &= (WPW^{-1})_+ W - WP \\ &= -(WPW^{-1})_- W. \end{aligned} \tag{1.5}$$

The evolution equation (1.4) is associated with an infinitesimal action ρ of $\mathcal{E}$ on the space $\mathcal{E}$. For $P \in \mathcal{E}$ the vector field $\rho(P)$ is given as follows.

$$W \longrightarrow -(WPW^{-1})_- W \in T_W \mathcal{E},$$

where the tangent space $T_W \mathcal{E}$ is identified with $\mathcal{E}$ by the structure of vector space of $\mathcal{E}$.

Theorem 1.3 For any $P, Q \in \mathcal{E}$ we have

$$\rho([P, Q]) = -[\rho(P), \rho(Q)].$$

Proof. We denote by ϵ_1 and ϵ_2 the time parameters with respect to P and Q, respectively. We set $\widetilde{P} = WPW^{-1}$ and $\widetilde{Q} = WQW^{-1}$. We have

$$\exp(\epsilon_1 \rho(P))W \equiv (1 - \epsilon_1 (\widetilde{P})_-)W \mod \epsilon_1^2,$$
$$\exp(\epsilon_2 \rho(Q))W \equiv (1 - \epsilon_2 (\widetilde{Q})_-)W \mod \epsilon_2^2.$$

Hence we have, modulo ϵ_1^2, ϵ_2^2

$$\begin{aligned}\exp(\epsilon_1\rho(P))(1-\epsilon_2(WQW^{-1})_-)&\\ &\equiv 1-\epsilon_2((1-\epsilon_1\widetilde{P}_-)\widetilde{Q}(1+\epsilon_1\widetilde{P}_-))_-\\ &\equiv 1-\epsilon_2\widetilde{Q}_-+\epsilon_1\epsilon_2[\widetilde{P}_-,\widetilde{Q}]_-.\end{aligned}$$

Thus we obtain, modulo ϵ_1^2, ϵ_2^2

$$\begin{aligned}\exp(\epsilon_1\rho(P))\exp(\epsilon_2\rho(Q))W&\\ &\equiv \exp(\epsilon_1\rho(P))(1-\epsilon_2(WQW^{-1})_-)W\\ &\equiv (1-\epsilon_2\widetilde{Q}_-+\epsilon_1\epsilon_2[\widetilde{P}_-,\widetilde{Q}]_-)(1-\epsilon_1\widetilde{P}_-)W\\ &\equiv (1-\epsilon_1\widetilde{P}_--\epsilon_2\widetilde{Q}_-+\epsilon_1\epsilon_2([\widetilde{P}_-,\widetilde{Q}]_-+\widetilde{Q}_-\widetilde{P}_-))W.\end{aligned}$$

Similarly we have, modulo ϵ_1^2, ϵ_2^2

$$\begin{aligned}\exp(\epsilon_2\rho(Q))\exp(\epsilon_1\rho(P))W&\\ &\equiv (1-\epsilon_2\widetilde{Q}_--\epsilon_1\widetilde{P}_-+\epsilon_1\epsilon_2([\widetilde{Q}_-,\widetilde{P}]_-+\widetilde{P}_-\widetilde{Q}_-))W.\end{aligned}$$

By the formula

$$\begin{aligned}\exp(\epsilon_1\rho(P))\exp(\epsilon_2\rho(Q))W-\exp(\epsilon_2\rho(Q))\exp(\epsilon_1\rho(P))W&\\ \equiv \epsilon_1\epsilon_2[\rho(P),\rho(Q)]W \qquad\qquad &\mod \epsilon_1^2,\ \epsilon_2^2,\end{aligned}$$

we obtain

$$\begin{aligned}[\rho(P),\rho(Q)]W &= \{[\widetilde{P}_-,\widetilde{Q}]_-+\widetilde{Q}_-\widetilde{P}_--[\widetilde{Q}_-,\widetilde{P}]_--\widetilde{P}_-\widetilde{Q}_-\}W\\ &= \{[\widetilde{P}_-,\widetilde{Q}_++\widetilde{Q}_-]_--[\widetilde{Q}_-,\widetilde{P}_++\widetilde{P}_-]_-+[\widetilde{Q}_-,\widetilde{P}_-]\}W.\end{aligned} \tag{1.6}$$

Since $[\widetilde{P}_+,\widetilde{Q}_+]_-=0$, the right hand side of (1.6) is equal to

$$[\widetilde{P},\widetilde{Q}]_-W=-\rho([P,Q])W \qquad \square$$

When $P, Q \in \mathcal{E}$ commute with each other, the following equations are compatible by Theorem 1.3:

$$\frac{\partial W}{\partial s} = (WPW^{-1})_+ W - WP,$$
$$\frac{\partial W}{\partial t} = (WQW^{-1})_+ W - WQ,$$

where $W = W(s,t)$.

In the case $r = 1$, for any $P \in \mathcal{E}$ and for any $I_0 = \mathcal{D}W(0)$ the solution I_t of the equation (1.3) satisfies the condition (1.1) for any t. With the choice $P = D_0^n$ $(n = 1, 2, \cdots)$, we obtain the KP-hierarchy ([7],[8]):

$$\frac{\partial W(t)}{\partial t} + W(t)D_0^n = (W(t)D_0^n W(t)^{-1})_+ W(t).$$

In higher dimensional case, we must choose a nice pair of the generator P and $I_0 = \mathcal{D}W(0)$ in order that I_t satisfies the condition (1.1) for any t. We shall see the evolution equation (1.4) constrains the initial value $W(0)$ in the following example.

Example 1.1 We consider (1.4) in the case $r = 2$. We take D_0^2 as the generator of the equation (1.4). We write

$$W(t) = \sum_{i+j \leq 0, j \geq 0} w_{i,j} D_0^i D_1^j, \quad w_{0,0} \equiv 1.$$

The operator $W(t)D_0^2$ is decomposed into the sum

$$\begin{array}{ccccc} \mathcal{E} & = & \mathcal{D}W & \oplus & \mathcal{E}_\phi \\ W(t)D_0^2 & = & (W(t)D_0^2 W(t)^{-1})_+ W & + & U \end{array} .$$

Then we have

$$(W(t)D_0^2 W(t)^{-1})_+ = (D_0^2 - 2\frac{\partial w_{-1,1}}{\partial x_0} D_1 - 2\frac{\partial w_{-1,0}}{\partial x_0}),$$

$$U = \sum_{i+j \leq 1, i<0} (-2\frac{\partial w_{i-1,j}}{\partial x_0} + 2\frac{\partial w_{-1,1}}{\partial x_0} w_{i,j-1} - \frac{\partial^2 w_{i,j}}{\partial x_0^2}$$
$$+2\frac{\partial w_{-1,0}}{\partial x_0} w_{i,j} + 2\frac{\partial w_{-1,1}}{\partial x_0}\frac{\partial w_{i,j}}{\partial x_1}) D_0^i D_1^j.$$

The equation (1.4) is equivalent to the following:

$$\frac{\partial w_{i,j}}{\partial t} + u_{i,j} = 0 \qquad \text{for } i+j \leq 0,$$
$$\frac{\partial w_{i-1,j}}{\partial x_0} - \frac{\partial w_{-1,1}}{\partial x_0} w_{i,j-1} = 0, \quad \text{for } i+j = 1.$$

The second equation constrains the initial data $W(0)$.

§2. Integrable systems in higher dimensions

In the example 1.1 we have considered the equation (1.3) for one generator $P = D_0^2$. In this section we will introduce a space V of generators, and determine the space of $\mathcal{D}$-submodules I of $\mathcal{E}$ such that the condition (1.1) is preserved under the time evolution (1.3) for any $P \in V$.

First we review two known examples, the self-dual Yang-Mills equations and the self-dual Einstein equations . We can interpret both the equations as integrable systems of three variables (see [14]).

Example 2.1 Self-dual Yang-Mills equations (see [15],[11]).

The self-dual Yang-Mills equations are written in the following form

$$\frac{\partial A_1}{\partial x_2}(x_1, x_2, s, t) = \frac{\partial A_2}{\partial x_1}(x_1, x_2, s, t),$$
$$[\frac{\partial}{\partial s} + A_1(x_1, x_2, s, t), \frac{\partial}{\partial t} + A_2(x_1, x_2, s, t)] = 0 \tag{2.1}$$

for gauge fields $A_1, A_2 \in \mathrm{Mat}(n \times n)$ on four-dimensional manifolds.

The evolution equation (1.4) is generalized to the case that W and P have matrix coefficients. We introduce the space $\mathcal{W}_{YM}(n)$ and the Lie algebra $V_{YM}(n)$:

$$\mathcal{W}_{YM}(n) = \{W(x_1, x_2, D_0) = \sum_{i \in \mathbb{N}} w_i(x_1, x_2) D_0^{-i};$$
$$w_i \in \mathrm{Mat}(n \times n, \mathbb{C}[[x_1, x_2]]),\ w_0 \equiv 1_n\}.$$

$$V_{YM}(n) = \{F_1(x_1,x_2,D_0)D_1 + F_2(x_1,x_2,D_0)D_2 + E(x_1,x_2,D_0);$$
$$F_1, F_2 \in \mathrm{Mat}(n\times n, \mathcal{D}),\ E \in \mathrm{Mat}(n\times n, \mathcal{E})\}.$$

The evolution equation (1.4) for $P \in V_{YM}(n)$ with any initial value $W \in \mathcal{W}_{YM}(n)$ has a solution in $\mathcal{W}_{YM}(n)$. We consider the equation (1.4) for $P = D_0D_1, D_0D_2 \in V_{YM}(n)$:

$$\begin{aligned}
&\frac{\partial W}{\partial s} + WD_0D_1 = (WD_0D_1W^{-1})_+W,\\
&\frac{\partial W}{\partial t} + WD_0D_2 = (WD_0D_2W^{-1})_+W.
\end{aligned} \tag{2.2}$$

In terms of the coefficients w_j of W the equation (2.2) is written in the form

$$\begin{aligned}
\frac{\partial w_i}{\partial s} &= \frac{\partial w_{i+1}}{\partial x_1} - \frac{\partial w_1}{\partial x_1}w_i,\\
\frac{\partial w_i}{\partial t} &= \frac{\partial w_{i+1}}{\partial x_2} - \frac{\partial w_1}{\partial x_2}w_i,
\end{aligned} \tag{2.3}$$

for $i > 0$. We set $A_j = \frac{\partial w_1}{\partial x_j}$ $(j = 1, 2)$. By eliminating w_2 from the equation (2.3) for $i = 1$, we obtain the equation (2.1).

Example 2.2 Self-dual Einstein equations (see [2],[12]).

The self-dual Einstein equations are written in the form (see [6])

$$\begin{aligned}
&\frac{\partial A_1}{\partial x_2}(x_1,x_2,s,t) = \frac{\partial A_2}{\partial x_1}(x_1,x_2,s,t),\\
&\frac{\partial B_1}{\partial x_2}(x_1,x_2,s,t) = \frac{\partial B_2}{\partial x_1}(x_1,x_2,s,t),\\
&[\frac{\partial}{\partial s} + A_1(x_1,x_2,s,t)\frac{\partial}{\partial x_1} + B_1(x_1,x_2,s,t)\frac{\partial}{\partial x_2},\\
&\qquad \frac{\partial}{\partial t} + A_2(x_1,x_2,s,t)\frac{\partial}{\partial x_1} + B_2(x_1,x_2,s,t)\frac{\partial}{\partial x_2}] = 0,\\
&\frac{\partial B_j}{\partial x_2}(x_1,x_2,s,t) + \frac{\partial A_j}{\partial x_1}(x_1,x_2,s,t) = 0 \quad (j = 1, 2).
\end{aligned} \tag{2.4}$$

In the following we forget the last equation in (2.4) for simplicity.

We introduce the space $\mathcal{W}_E$ and the Lie algebra V_E:

$$\mathcal{W}_E = \{W = \sum_{j,k\in\mathbb{N}} \frac{1}{j!k!} G_1^j G_2^k D_1^j D_2^k; \quad G_i = \sum_{j<0} g_{i,j}(x_1,x_2) D_0^j (i=1,2)\}.$$

$$V_E = \{F_1(x_1,x_2,D_0)D_1 + F_2(x_1,x_2,D_0)D_2;\ F_1,F_2 \in \mathcal{E}\}.$$

The evolution equation (1.4) for $P \in V_E$ with any initial value $W \in \mathcal{W}_E$ has a solution in $\mathcal{W}_E$. We consider (1.4) for $P = D_0D_1, D_0D_2 \in V_E$:

$$\begin{aligned} \frac{\partial W}{\partial s} + WD_0D_1 &= (WD_0D_1W^{-1})_+W, \\ \frac{\partial W}{\partial t} + WD_0D_2 &= (WD_0D_2W^{-1})_+W. \end{aligned} \tag{2.5}$$

In terms of the coefficients $g_{i,j}$ the equation (2.5) is written in the form:

$$\begin{aligned} \frac{\partial g_{i,j}}{\partial s} &= \frac{\partial g_{i,j-1}}{\partial x_1} - \frac{\partial g_{1,-1}}{\partial x_1}\frac{\partial g_{i,j}}{\partial x_1} - \frac{\partial g_{2,-1}}{\partial x_1}\frac{\partial g_{i,j}}{\partial x_2}, \\ \frac{\partial g_{i,j}}{\partial t} &= \frac{\partial g_{i,j-1}}{\partial x_2} - \frac{\partial g_{1,-1}}{\partial x_2}\frac{\partial g_{i,j}}{\partial x_1} - \frac{\partial g_{2,-1}}{\partial x_2}\frac{\partial g_{i,j}}{\partial x_2} \quad \text{for } i=1,2,\ j<0. \end{aligned} \tag{2.6}$$

We set $A_j = -\frac{\partial g_{1,-1}}{\partial x_j}, B_j = -\frac{\partial g_{2,-1}}{\partial x_j}\,(j=1,2)$. By eliminating $g_{i,-2}\,(i = 1,2)$ from the equation (2.6) for $i=1,2,\ j=-1$, we obtain the equation (2.4).

We shall unify these two examples and obtain more general systems. We introduce the Lie subalgebra V of $\mathcal{E}$ which contains both the Lie algebras V_{YM} and V_E:

$$V = \{F_0(x',D_0)x_0 + \sum_{0<j<r} F_j(x',D_0)D_j + E(x',D_0); \\ F_k(x',D_0),\ E(x',D_0) \in \mathcal{E},\ (0 \le k < r)\},$$

where $x' = (x_1, x_2, \cdots, x_{r-1})$.

Lemma 2.1 V is isomorphic to $\mathcal{D}(1) \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$ as a Lie algebra.

Proof. Set $X = \mathbb{C}^r = \{(z_0, z_1, \cdots, z_{r-1}) \in \mathbb{C}^r\}$, and $Y = \{z \in X; z_0 = 0\}$. We take the transformation

$$x_0 = z_0^2 D_{z_0},\ D_{x_0} = z_0^{-1},\ x' = z',\ D_{x'} = D_{z'}.$$

Then the transform of V is $\mathcal{D}(1) \otimes \mathbb{C}[[z]][z_0^{-1}]$. $\square$

The Lie algebra $\mathcal{D}(1) \otimes \mathbb{C}[[x]][x_0^{-1}]$ is the direct sum of the Lie algebra $\Theta \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$ of vector fields and the commutative Lie algebra $\mathbb{C}[[x]][x_0^{-1}]$, where the Lie algebra Θ is defined by

$$\Theta = \sum_{0 \leq j < r} \mathcal{O} D_j.$$

The Lie algebra $\Theta \otimes_{\mathcal{O}} \mathbb{C}[[x]][x_0^{-1}]$ corresponds to the infinitesimal coordinate transformations, and the Lie algebra $\mathbb{C}[[x]][x_0^{-1}]$ corresponds to the infinitesimal gauge transformations of a line bundle.

Now we shall determine the set of $\mathcal{D}$-submodules in $\mathcal{E}$ such that the condition (1.1) is preserved under (1.3) with respect to any $P \in V$. We introduce the subspaces $\mathcal{W}_m$ and $\mathcal{W}_{YM}$ of $\mathcal{E}$:

$$\begin{aligned}
\mathcal{W}_m = \{W(x, D_x) = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_x^\alpha \in \mathcal{E};& \\
G^\alpha = G_0^{\alpha_0} G_1^{\alpha_1} \cdots\ G_{r-1}^{\alpha_{r-1}} \quad \text{where } G_j = G_j(x', D_0) \in \mathcal{E}(-1)\},& \\
\mathcal{W}_{YM} = \{W(x', D_0) = \sum_{i \in \mathbb{N}} w_i(x') D_0^{-i};\ w_i \in \mathbb{C}[[x_1, \cdots, x_{r-1}]],\ w_0 \equiv 1.\}.&
\end{aligned}$$

Proposition 2.2 Let $I_0 = \mathcal{D} W_0$ be a $\mathcal{D}$-submodule of $\mathcal{E}$. Assume that the time evolution of I_0 for any $P \in V$ also satisfies the condition (1.1). Then W_0 factorizes into the product of $W_m \in \mathcal{W}_m$ and $W_{YM} \in \mathcal{W}_{YM}$;

$$W_0 = W_m W_{YM}.$$

Remark. The operator W_m corresponds to the equation of self-dual metrics (see §3).

Proof of Proposition 2.2.

Step 1.

Lemma 2.3 Let $W(t)$ be the solution of (1.4) for $P \in V$ with $W(0) = W_0$. Then $(W(t)PW(t)^{-1})_-$ is contained in $\mathcal{E}_\phi(0)$.

Proof.

It follows from the equation (1.5) that

$$\frac{\partial W(t)}{\partial t} W(t)^{-1} = -(W(t)PW(t)^{-1})_-.$$

Since the operator $\frac{\partial W(t)}{\partial t} W(t)^{-1}$ is contained in $\mathcal{E}(0)$, we obtain Lemma 2.3. □

Step 2. For any $P \in \mathcal{E}$, let $\widetilde{P}$ be $W_0 P W_0^{-1}$.

Lemma 2.4 Suppose that $P, Q \in V$ commute with each other. The operator $[\widetilde{P}_+, \widetilde{Q}_-]_-$ is contained in $\mathcal{E}_\phi(0)$.

Proof. We consider the solution $W = W(s,t)$ of the equation (1.5) for the operators P, Q:

$$\begin{aligned} \frac{\partial W}{\partial s} &= -(WPW^{-1})_- W, \\ \frac{\partial W}{\partial t} &= -(WQW^{-1})_- W, \end{aligned} \tag{2.7}$$

where $W = W(s,t)$. Since $[P,Q] = 0$, the system (2.7) is compatible by Theorem 1.3. It follows from Lemma 2.3 that WPW^{-1} is contained in $\mathcal{D} + \mathcal{E}_\phi(0)$. We have

$$\begin{aligned} \frac{\partial}{\partial t}(WPW^{-1}) &= \frac{\partial W}{\partial t} PW^{-1} - WPW^{-1}\frac{\partial W}{\partial t} W^{-1} \\ &= -(WQW^{-1})_- WPW^{-1} + WPW^{-1}(WQW^{-1})_- \\ &= [WPW^{-1}, (WQW^{-1})_-]. \end{aligned}$$

Hence $[\widetilde{P}, \widetilde{Q}_-]$ is contained in $\mathcal{D} + \mathcal{E}_\phi(0)$. Since $[\widetilde{P}_-, \widetilde{Q}_-]$ is contained in $\mathcal{E}_\phi(0)$ by Lemma 2.3, $[\widetilde{P}_+, \widetilde{Q}_-]$ is contained in $\mathcal{D} + \mathcal{E}_\phi(0)$. $\square$

Step 3 Recall that $\widetilde{P}$ denotes $W_0 P W_0^{-1}$ for any $P \in \mathcal{E}$. We define the operators $G_i, H_i \in \mathcal{E}$ $(0 \leq i < r)$ as follows:

$$
\begin{aligned}
\widetilde{D}_0 &= D_0 - G_0(x, D_x), \\
\widetilde{x}_i &= x_i + G_i(x, D_x) \quad \text{for } i = 1, 2, \cdots, r-1, \\
\widetilde{x}_0 &= x_0 + H_0(x, D_x), \\
\widetilde{D}_i &= D_i + H_i(x, D_x) \quad \text{for } i = 1, 2, \cdots, r-1.
\end{aligned} \tag{2.8}
$$

Lemma 2.5 Assume that W_0 satisfies the condition in Proposition 2.2. The operators G_i, H_i are written in the following form:

$$
\begin{aligned}
G_i &= G_i(x', D_0) \in \mathcal{E}(-1), \\
H_i &= \sum_{0<j<r} K_{ij}(x', D_0) D_j + L_i(x, D_0) \quad \text{for } 0 \leq i < r, \\
&\qquad \text{where} \quad K_{ij}(x', D_0) \in \mathcal{E}, \quad L_i(x, D_0) \in \mathcal{E}(-1), \\
&\qquad \text{and} \quad x' = (x_1, \cdots, x_{r-1}).
\end{aligned}
$$

Proof. We shall show the following statement $(2.9)_n$ by the induction on n:

$$
\begin{aligned}
& G_i = G_i' + G_i'', \quad H_i = H_i' + H_i'' \\
& \text{for some } G_i', H_i', G_i'' \text{ and } H_i'' \ (0 \leq i < r) \text{ such that} \\
& G_0'', H_i'' \in \mathcal{E}(-n), \quad G_i'', H_0'' \in \mathcal{E}(-n-1) \quad \text{for } 0 < i < r, \\
& G_i' = G_i'(x', D_0) \in \mathcal{E}(-1), \\
& H_i' = \sum_{0<j<r} K_{ij}(x', D_0) D_j + L_i(x, D_0) \quad \text{for } 0 \leq i < r \\
& \text{with} \quad K_{ij}(x', D_0),\ L_i(x, D_0) \in \mathcal{E}(-1).
\end{aligned} \tag*{$(2.9)_n$}
$$

It is evident that the statement $(2.9)_0$ is true. By assuming $(2.9)_n$ we shall prove $(2.9)_{n+1}$.

We expand the operators:

$$
\begin{aligned}
G_j'' &= \sum_{k\leq 0, \alpha'\in\mathbb{N}^{r-1}} g_{k,\alpha'}^{(j)} D_0^k D_{x'}^{\alpha'}, \\
H_j'' &= \sum_{k\leq 0, \alpha'\in\mathbb{N}^{r-1}} h_{k,\alpha'}^{(j)} D_0^k D_{x'}^{\alpha'}.
\end{aligned}
\tag{2.10}
$$

Since G_0'' belongs to $\mathcal{E}(-n)$, we have

$$
\begin{aligned}
\widetilde{D}_0^{n+2} &= (D_0 - G_0' - G_0'')^{n+2} \\
&\equiv (D_0 - G_0')^{n+2} - (n+2)G_0''(D_0 - G_0')^{n+1} \\
&\equiv (D_0 - G_0')^{n+2} - (n+2)G_0'' D_0^{n+1} \quad \text{modulo } \mathcal{E}(0).
\end{aligned}
$$

The operator $(D_0 - G_0')^{n+2}$ belongs to $\mathcal{D}+\mathcal{E}(0)$, because G_0' does not contain $D_j (0 < j < r)$. Hence we obtain

$$
\widetilde{D}_0^{n+2} \equiv -(n+2)G_0'' D_0^{n+1} \quad \text{modulo } \mathcal{D}+\mathcal{E}(0).
$$

Since $\widetilde{D}_0^{n+2}$ is contained in $\mathcal{D}+\mathcal{E}(0)$ for any $n \in \mathbb{Z}$ by Lemma 2.3, $G_0'' D_0^{n+1}$ is also contained in $\mathcal{D}+\mathcal{E}(0)$. Therefore we obtain

$$
g_{-n-|\alpha'|,\alpha'}^{(0)} = 0 \quad \text{for} \quad |\alpha'| \geq 2. \tag{2.11}
$$

Similarly we have, modulo $\mathcal{D}+\mathcal{E}(0)$

$$
\begin{aligned}
\widetilde{D}_0^{n+1}\widetilde{D}_j &\equiv -(n+1)G_0'' D_0^n D_j + H_j'' D_0^{n+1}, \\
\widetilde{x}_0 \widetilde{D}_0^{n+2} &\equiv -(n+2)x_0 G_0'' D_0^{n+1} + H_0'' D_0^{n+2}, \\
\widetilde{x}_j \widetilde{D}_0^{n+1}\widetilde{D}_k &\equiv G_j'' D_0^{n+1} D_k + x_j(H_k'' D_0^{n+1} - (n+1)G_0'' D_0^n D_k).
\end{aligned}
$$

By Lemma 2.3, $\widetilde{D}_0^{n+1}\widetilde{D}_j, \widetilde{x}_0\widetilde{D}_0^{n+2}$ and $\widetilde{x}_j\widetilde{D}_0^{n+1}\widetilde{D}_k$ are contained in $\mathcal{D}+\mathcal{E}(0)$. Thus we obtain

$$
\begin{aligned}
h^{(j)}_{-|\alpha'|-n,\alpha'} &= 0 \quad \text{for } |\alpha'| \geq 3,\\
h^{(j)}_{-2-n,\alpha'+e_j} &=(n+1)g^{(0)}_{-1-n,\alpha'} \quad \text{for } |\alpha'| = 1,\\
h^{(0)}_{-|\alpha'|-n-1,\alpha'} &= 0 \quad \text{for } |\alpha'| \geq 2,\\
g^{(j)}_{-|\alpha'|-n-1,\alpha'} &= 0 \quad \text{for } |\alpha'| \geq 2,
\end{aligned}
\tag{2.12}
$$

where $j = 1, 2, \cdots, r-1$ and $e_j = (0, \cdots, \overset{\overset{j}{\vee}}{1}, \cdots, 0) \in \mathbb{N}^{r-1}$.

We denote by $\mathcal{V}$ and $\mathcal{V}'$ the subspaces of $\mathcal{D}+\mathcal{E}(0)$:

$$
\begin{aligned}
\mathcal{V} &= \{ \sum_{0<k<r} F_k(x, D_0)D_k + E(x, D_0); \quad F_k(x, D_0), E(x, D_0) \in \mathcal{E}\},\\
\mathcal{V}' &= \{ \sum_{0<k<r} F_k(x', D_0)D_k + E(x, D_0); \quad F_k(x', D_0), E(x, D_0) \in \mathcal{E}\}.
\end{aligned}
$$

For $P, Q \in \mathcal{V}'$ the commutator $[P, Q]$ is contained in $\mathcal{V}$. Since H'_j $(0 \leq j < r)$ are contained in $\mathcal{V}'$, $[H'_i, H'_j]$ $[x_0, H'_j]$ $(0 \leq i, j < r)$ are contained in $\mathcal{V}$.

For $j > 0$ we have

$$
\begin{aligned}
[\widetilde{x}_0, \widetilde{D}_j] &=[x_0, H_j] - \frac{\partial H_0}{\partial x_j} + [H_0, H_j]\\
&=[x_0, H''_j] - \frac{\partial H''_0}{\partial x_j} + ([H''_0, H''_j] + [H'_0, H''_j] + [H''_0, H'_j])\\
&\qquad\qquad + ([x_0, H'_j] - \frac{\partial H'_0}{\partial x_j} + [H'_0, H'_j])\\
&\equiv[x_0, H''_j] - \frac{\partial H''_0}{\partial x_j} \qquad \text{modulo } \mathcal{E}(-n-2) + \mathcal{V}.
\end{aligned}
$$

Since $[\widetilde{x}_0, \widetilde{D}_j] = 0$, $[x_0, H''_j] - \frac{\partial H''_0}{\partial x_j}$ belongs to $\mathcal{E}(-n-2)+\mathcal{V}$. By (2.10) we have, modulo $\mathcal{E}(-n-2)+\mathcal{V}$

$$
\begin{aligned}
&[x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} \\
&\qquad \equiv \sum_{|\alpha'|\geq 2} ((n+|\alpha'|)h^{(j)}_{-n-|\alpha'|,\alpha'} - \frac{\partial h^{(0)}_{-n-1-|\alpha'|,\alpha'}}{\partial x_j}) D_0^{-1-n-|\alpha'|} D_{x'}^{\alpha'}
\end{aligned} \tag{2.13}
$$

We obtain from (2.12) and (2.13) that

$$
\begin{aligned}
h^{(j)}_{-2-n,\alpha'} &= 0 \quad \text{for } |\alpha'| = 2, \quad 0 < j < r, \\
g^{(0)}_{-1-n,\alpha'} &= 0 \quad \text{for } |\alpha'| = 1.
\end{aligned} \tag{2.14}
$$

Similarly by the equations

$$
\begin{aligned}
&[\widetilde{D}_0, \widetilde{D}_j] = \frac{\partial H_j}{\partial x_0} + \frac{\partial G_0}{\partial x_j} - [G_0, H_j] = 0 \quad \text{for} \quad 0 < j < r, \\
&[\widetilde{D}_0, \widetilde{x}_j] = \frac{\partial G_j}{\partial x_0} + [G_j, G_0] - [G_0, x_j] = 0 \quad \text{for} \quad 0 < j < r, \\
&[\widetilde{D}_0, \widetilde{x}_0] = 1 + [D_0, H_0] - [G_0, x_0] + [G_0, H_0] = 1,
\end{aligned}
$$

we obtain

$$
\frac{\partial h^{(j)}_{-1-n,e_k}}{\partial x_0} = 0, \quad \frac{\partial h^{(0)}_{-2-n,e_k}}{\partial x_0} = 0, \quad \frac{\partial g^{(j)}_{-1-n,0}}{\partial x_0} = 0. \tag{2.15}
$$

For $0 < i < r$ we have modulo $\mathcal{E}(-1)$

$$
\begin{aligned}
\widetilde{D}_0^n \widetilde{D}_i &= (D_0 - G_0' - G_0'')^n (D_i + H_i' + H_i'') \\
&\equiv ((D_0 - G_0')^n - nG_0''(D_0 - G_0')^{n-1})(D_i + H_i' + H_i'') \\
&\equiv -nG_0'' D_0^{n-1} D_i + (D_0 - G_0')^n (D_i + H_i') + H_i'' D_0^n.
\end{aligned} \tag{2.16}
$$

By (2.12) and (2.14) we have

$$H_i'' D_0^n \equiv \sum_{0<j<r} h^{(i)}_{-1-n,e_j} D_0^{-1} D_j \quad \text{modulo } \mathcal{E}(-1),$$

$$-nG_0'' D_0^{n-1} D_i \equiv -ng^{(0)}_{-n,0} D_0^{-1} D_i \quad \text{modulo } \mathcal{E}(-1).$$

We set K_{ij} and L_i in $(2.9)_n$ as follows:

$$K_{ij}(x', D_0) = \sum_{m<0} k_{i,j,m}(x') D_0^m,$$

$$L_i(x, D_0) = \sum_{m<0} l_{i,m}(x') D_0^m$$

By $(2.9)_n$ we have

$$\begin{aligned}(\widetilde{D}_0 \widetilde{D}_i)_+ &= (D_0 - G_0)(D_i + H_i' + H_i'') \\ &= D_0 D_i + \{D_0(H_i' + \sum_{1\le j<r} K_{ij} + L_i)\}_+ \\ &= D_0 D_i + \sum_{1\le j<r} (h^{(i)}_{-1,e_j} + k_{i,j,-1}) D_j + l_{i,-1}\end{aligned} \tag{2.17}$$

Sublemma. We have

$$\frac{\partial g^{(0)}_{-n,0}}{\partial x_0} = 0. \tag{2.18}$$

Proof. The commutator $[(\widetilde{D}_0 \widetilde{D}_i)_+, \widetilde{D}_0^n \widetilde{D}_i]$ is contained in $\mathcal{D}+\mathcal{E}(0)$ by Lemma 2.4. By (2.16) and (2.17) we have, modulo $\mathcal{E}(0)$

$$\begin{aligned}&[(\widetilde{D}_0 \widetilde{D}_i)_+, \widetilde{D}_0^n \widetilde{D}_i] \\ \equiv&[(\widetilde{D}_0 \widetilde{D}_i)_+, -ng^{(0)}_{-n,0} D_0^{-1} D_i \\ &\qquad + \sum_{0<j<r} h^{(i)}_{-1-n,e_j} D_0^{-1} D_j + (D_0 - G_0')^n (D_i + H_i')] \\ \equiv&[D_0 D_i, -ng^{(0)}_{-n,0} D_0^{-1} D_i] \\ &\quad + [(\widetilde{D}_0 \widetilde{D}_i)_+, \sum_{0<j<r} h^{(i)}_{-1-n,e_j} D_0^{-1} D_j + (D_0 - G_0')^n (D_i + H_i')].\end{aligned}$$

Since $(\widetilde{D}_0\widetilde{D}_i)_+$ and $\sum_{0<j<r} h^{(i)}_{-1-n,e_j} D_0^{-1}D_j + (D_0 - G_0')^n(D_i + H_i')$ are contained in $\mathcal{V}$, the operator

$$
\begin{aligned}
&[D_0D_i, -ng^{(0)}_{-n,0}D_0^{-1}D_i] \\
&\qquad = -n\frac{\partial g^{(0)}_{-n,0}}{\partial x_0}D_0^{-1}D_i^2 - n\frac{\partial g^{(0)}_{-n,0}}{\partial x_i}D_i - n\frac{\partial^2 g^{(0)}_{-n,0}}{\partial x_0 \partial x_i}
\end{aligned}
$$

is contained in $\mathcal{D}+\mathcal{E}(0)$. This implies (2.18). □

It follows from (2.11), (2.12) and (2.14) that we have

$$
\begin{aligned}
G_0' - g^{(0)}_{-n,0}D_0^{-n} &\in \mathcal{E}(-n-1), \\
G_i' - g^{(i)}_{-n-1,0}D_0^{-n-1} &\in \mathcal{E}(-n-2), \\
H_0' - (\sum_{0<k<r} h^{(0)}_{-n-2,e_k}D_0^{-n-2}D_k + h^{(0)}_{-n-1,0}D_0^{-n-1}) &\in \mathcal{E}(-n-2), \\
H_i' - (\sum_{0<k<r} h^{(i)}_{-n-1,e_k}D_0^{-n-1}D_k + h^{(i)}_{-n,0}D_0^{-n}) &\in \mathcal{E}(-n-1).
\end{aligned}
$$

for $0 < i < r$. Thus we obtain $(2.9)_{n+1}$ from (2.15) and (2.18). □

Step 4 Now we shall prove Proposition 2.2. We introduce a micro-differential operator

$$
W_m = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'},
$$

where $G^\alpha = G_0^{\alpha_0}G_1^{\alpha_1}\cdots G_{r-1}^{\alpha_{r-1}}$ is given in Lemma 2.5. Then we have

$$
\begin{aligned}
[D_0, W_m] &= \sum_{\alpha \in \mathbb{N}^r} \frac{\alpha_0}{\alpha!} G^\alpha x_0^{\alpha_0 - 1} D_{x'}^{\alpha'} \\
&= G_0 \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'} = G_0 W_m.
\end{aligned}
$$

Hence we get

$$W_m D_0 W_m^{-1} = D_0 - G_0. \tag{2.20}$$

Similarly we obtain that

$$W_m x_j W_m^{-1} = x_j + G_j \quad (1 \leq j < r). \tag{2.21}$$

Set $W_{YM} = W_m^{-1} W$. Then it follows from (2.8), (2.20) and (2.21) that W_{YM} commutes with $D_0, x_1, \cdots, x_{r-1}$. Therefore W_{YM} is contained in $\mathcal{W}_{YM}$.

Thus we have completed the proof of Proposition 2.2. $\square$

We set

$$\mathcal{W} = \{W_m W_{YM}; \quad W_m \in \mathcal{W}_m, W_{YM} \in \mathcal{W}_{YM}\}.$$

We shall investigate the structure of $\mathcal{W}, \mathcal{W}_m$ and $\mathcal{W}_{YM}$.

Proposition 2.6 The spaces $\mathcal{W}$, $\mathcal{W}_m$ and $\mathcal{W}_{YM}$ are groups by the composition of microdifferential operators, and $\mathcal{W}_{YM}$ is a normal subgroup in $\mathcal{W}$.

Proof. It is evident that $\mathcal{W}_{YM}$ is an abelian group.

Let $W_m = \sum_\alpha \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'}$ and $W'_m = \sum_\beta \frac{1}{\beta!} F^\beta x_0^{\beta_0} D_{x'}^{\beta'}$ be operators in $\mathcal{W}_m$. For any microdifferential operator $P(x', D_0)$, we set

$$\widetilde{P} := W_m P W_m^{-1} = P(x_1 + G_1, \cdots, x_{r-1} + G_{r-1}, D_0 - G_0).$$

The last equality follows from (2.20) and (2.21). Noting that $\widetilde{P}$ commutes with G_j, the composition

$$\begin{aligned} W_m W'_m &= \sum_\gamma \frac{1}{\beta!} \widetilde{F}^\beta W_m x_0^{\beta_0} D_{x'}^{\beta'} \\ &= \sum_{\alpha,\beta} \frac{1}{\beta!} \widetilde{F}^\beta \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0+\beta_0} D_{x'}^{\alpha'+\beta'} \\ &= \sum_\gamma \frac{1}{\gamma!} (\widetilde{F} + G)^\gamma x_0^{\gamma_0} D_{x'}^{\gamma'} \end{aligned} \tag{2.22}$$

is contained in $\mathcal{W}_m$. For $W_{YM} = \sum_i w_i(x')D_0^{-i} \in \mathcal{W}_{YM}$ the operator

$$\begin{aligned} & W_m W_{YM} W_m^{-1} \\ =& \sum_{i\geq 0} w_i(x_1 + G_1, x_2 + G_2, \cdot \quad , x_{r-1} + G_{r-1})(D_0 - G_0)^{-i} \end{aligned} \tag{2.23}$$

is contained in $\mathcal{W}_{YM}$. Since $\mathcal{W} = \mathcal{W}_m \mathcal{W}_{YM}$, it follows from (2.22) and (2.23) that $\mathcal{W}$ is a group and that $\mathcal{W}_{YM}$ is a normal subgroup of $\mathcal{W}$. $\square$

We define the Lie subalgebras V_m and V_{YM} of V

$$\begin{aligned} V_m = \{F_0(x', D_0)x_0 + \sum_{0<j<r} & F_j(x', D_0)D_j; \\ & F_k(x', D_0) \in \mathcal{E}, \ (0 \leq k < r)\}, \\ V_{YM} = \{E(x', D_0); \ E(x', D_0) & \in \mathcal{E}\}. \end{aligned}$$

We have

$$V = V_m \oplus V_{YM}.$$

Proposition 2.7 For any $P \in V$ (resp. V_m, V_{YM}) and $W \in \mathcal{W}$ (resp. $\mathcal{W}_m, \mathcal{W}_{YM}$), we have $WPW^{-1} \in V$ (resp. V_m, V_{YM}).

Proof. For $W = \sum_\alpha \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'} \in \mathcal{W}_m$ we have

$$\begin{aligned} [D_i, W] &= \sum_{0\leq j<r} \sum_\alpha \frac{\alpha_j}{\alpha!} G^\alpha \frac{\partial G_j}{\partial x_i} x_0^{\alpha_0} D_{x'}^{\alpha'} \\ &= \sum_{0<j<r} \frac{\partial G_j}{\partial x_i} W D_j + \frac{\partial G_0}{\partial x_i} W x_0 \end{aligned}$$

for any $i = 1, 2, \cdots, r-1$. Similarly we have

$$[x_0, W] = \sum_{0<j<r} [x_0, G_j] W D_j + [x_0, G_0] W x_0.$$

Hence we obtain

$$
\begin{aligned}
WD_iW^{-1} &= D_i - \sum_{0<j<r} \frac{\partial G_j}{\partial x_i} WD_jW^{-1} - \frac{\partial G_0}{\partial x_i} Wx_0W^{-1} \quad \text{for } 0 < i < r, \\
Wx_0W^{-1} &= x_0 - \sum_{0<j<r} [x_0, G_j] WD_jW^{-1} - [x_0, G_0] Wx_0W^{-1}.
\end{aligned} \tag{2.24}
$$

We set

$$
G_{ij} = \begin{cases} \frac{\partial G_j}{\partial x_i} & \text{for } 0 < i < r, \\ [x_0, G_j] & i = 0, \end{cases}
$$

for $0 \leq j < r$. Then we have

$$
\begin{aligned}
x_0 &= (1 + G_{00}) Wx_0W^{-1} + \sum_{0<j<r} G_{0j} WD_jW^{-1}, \\
D_i &= G_{i0} Wx_0W^{-1} + \sum_{0<j<r} (\delta_{ij} + G_{ij}) WD_jW^{-1} \quad \text{for} \quad 0 < i < r.
\end{aligned} \tag{2.25}
$$

Since $G_{ij} \in \mathcal{E}(-1)$, The matrix $(\delta_{ij} + G_{ij})_{0 \leq i,j < r}$ is invertible. Hence $WD_iW^{-1}\,(0 < i < r)$ and Wx_0W^{-1} are contained in V_m, because G_{ij} is independent of $x_0, D_1, \cdots, D_{r-1}$. For any operator $P = P(x', D_0) \in \mathcal{E}$, WPW^{-1} is independent of $x_0, D_1, \cdots, D_{r-1}$ by (2.20) and (2.21). Hence we get Proposition 2.7 for V_m and $\mathcal{W}_m$.

The proposition is evident for V_{YM} and $\mathcal{W}_{YM}$. For $W_m \in \mathcal{W}_m$ and $E \in V_{YM}$ the operator $W_m E W_m^{-1}$ is contained in V_{YM} by (2.20) and (2.21). For $W_{YM} \in \mathcal{W}_{YM}$ we have

$$
\begin{aligned}
W_{YM} D_i W_{YM}^{-1} &= D_i - \frac{\partial W_{YM}}{\partial x_i} W_{YM}^{-1} \quad \text{for } 0 < i < r, \\
W_{YM} x_0 W_{YM}^{-1} &= x_0 - [x_0, W_{YM}] W_{YM}^{-1}.
\end{aligned}
$$

Since W_{YM} commutes with $D_0, x_1, \cdots, x_{r-1}$, the operator $W_{YM} P W_{YM}^{-1}$ is contained in V for any $P \in V$. Since $V = V_m \oplus V_{YM}$ and $\mathcal{W} = \mathcal{W}_m \mathcal{W}_{YM}$, we get Proposition 2.7. □

Example 2.3 We shall write down the evolution equations (1.4) for $W \in \mathcal{W}$ in the case $r = 3$. We take $D_0 D_1, D_0 D_2$ as generators of (1.4).

We set

$$W = \Big(\sum_{\alpha\in\mathbb{N}^3} \frac{1}{\alpha!} G^\alpha x_0^{\alpha_0} D_{x'}^{\alpha'}\Big)\Big(\sum_i w_i(x') D_0^{-i}\Big),$$
$$\text{where}\quad G_i(x_1,x_2,D_0) = \sum_{j<0} g_{i,j}(x_1,x_2) D_0^j \quad (i=1,2).$$

Then we have

$$(\widetilde{D}_0\widetilde{D}_1)_+ = D_0D_1 - \frac{\partial g_{0,-1}}{\partial x_1}x_0 - \sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_1}D_i - \frac{\partial w_1}{\partial x_1},$$
$$(\widetilde{D}_0\widetilde{D}_2)_+ = D_0D_2 - \frac{\partial g_{0,-1}}{\partial x_2}x_0 - \sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_2}D_i - \frac{\partial w_1}{\partial x_2}.$$

Taking time parameters s and t with respect to D_0D_1 and D_0D_2, respectively, we obtain the evolution equations

$$\frac{\partial W}{\partial s} + WD_0D_1 = \Big(D_0D_1 - \frac{\partial g_{0,-1}}{\partial x_1}x_0 - \sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_1}D_i - \frac{\partial w_1}{\partial x_1}\Big)W,$$
$$\frac{\partial W}{\partial t} + WD_0D_2 = \Big(D_0D_2 - \frac{\partial g_{0,-1}}{\partial x_2}x_0 - \sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_2}D_i - \frac{\partial w_1}{\partial x_2}\Big)W. \tag{2.26}$$

It follows from (2.26) that we obtain the Zakharov-Shabat type equation

$$\Big[\frac{\partial}{\partial s} - D_0D_1 + \frac{\partial g_{0,-1}}{\partial x_1}x_0 + \sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_1}D_i + \frac{\partial w_1}{\partial x_1},$$
$$\frac{\partial}{\partial t} - D_0D_2 + \frac{\partial g_{0,-1}}{\partial x_2}x_0 + \sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_2}D_i + \frac{\partial w_1}{\partial x_2}\Big] = 0. \tag{2.27}$$

We shall investigate the infinitesimal action ρ of the Lie subalgebra V of $\mathcal{E}$. Remark that the Lie algebra of the group $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$) is canonically isomorphic to the Lie subalgebra $V_- = V \cap \mathcal{E}_\phi$ (resp. $(V_m)_-$, $(V_{YM})_-$).

Theorem 2.8 The action ρ of V (resp. V_m, V_{YM}) preserves the space $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$). The action of V_- (resp. $(V_m)_-$, $(V_{YM})_-$) coincides with the infinitesimal right action of on the group $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$).

Proof. For any element $P \in V$ (resp. V_m, V_{YM}) and any operator $W \in \mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$), we have

$$\rho(P) = -(WPW^{-1})_- W \in T_W \mathcal{E}.$$

By Proposition 2.7, $-(WPW^{-1})_- W$ is contained in the tangent space of $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$) at W. Taking $P \in V_-$ (resp. $(V_m)_-$, $(V_{YM})_-$), we have $\rho(P) = -WP$. Hence the action ρ is the right action of vector fields. □

By Theorem 2.8 the Lie algebra V (resp. V_m, V_{YM}) acts on $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$) transitively.

§3. Twistor theory and integrable systems

On oriented Riemannian manifolds of dimension four, the Weyl curvature tensor C decomposes into two components, the self-dual part C_+ and the anti-self-dual part C_-. A manifold is called self-dual (resp. anti-self-dual) when C_- (resp. C_+) vanishes . Penrose [5] showed that the vanishing of the anti-self-dual part of the Weyl tensor is precisely the integrability condition of the existence of a curved twistor space.

In this section we prove that the equation $C_+=0$ is the compatibility condition of the deformation equations of filtered $\mathcal{D}$-submodules in $\mathcal{E}$ (See [13] in which the Frobenius integrability condition of the equations of self dual metrics is discussed). We get the equations of self dual metrics from the equation (2.26) for $W \in \mathcal{W}_m$.

Let M be a complex four-manifold and g a holomorphic metric, i.e. a non-degenerate symmetric holomorphic covariant two-tensor on M. We shall choose a holomorphic orientation on M which is necessary to define the complex Hodge *-operator. Our discussion being only local, we can assume the existence of two complex vector bundles S_+ and S_-: the bundles of self-dual and anti-self-dual spinors.

Let $\{e_j\}_{j=1,2,3,4}$ denote a local coframe on M such that $g = e_1e_2+e_3e_4$. We can write them in spinor language as

$$\begin{bmatrix} e_4 & e_2 \\ -e_1 & e_3 \end{bmatrix} = \begin{bmatrix} \psi_1\phi_1 & \psi_1\phi_2 \\ \psi_2\phi_1 & \psi_2\phi_2 \end{bmatrix} \tag{3.1}$$

where ψ_1, ψ_2 (resp. ϕ_1, ϕ_2) are the bases of self-dual (resp. anti-self-dual) spinor coframes.

We take $P = P(S_-)$, the projective bundle of the rank two vector bundle S_-. We parametrize S_- locally by

$$(x, \mu_1, \mu_2) \longrightarrow \mu_1\phi_1(x) + \mu_2\phi_2(x),$$

and $\mu = \mu_1/\mu_2$ is an affine coordinate for $\mu_2 \neq 0$.

Theorem 3.1 ([5]) The Riemannian manifold (M, g) is self-dual iff the following Pfaffian system Ω on P is integrable:

$$\Omega: \quad \begin{aligned} &\theta := d\mu + \omega_{21}\mu^2 - (\omega_{22} - \omega_{11})\mu - \omega_{12} = 0, \\ &\sigma_1 := \mu e_4 + e_2 = 0, \\ &\sigma_2 := -\mu e_1 + e_4 = 0, \end{aligned} \tag{3.2}$$

where ω_{ij} is the connection form of S_- with respect to the frame ϕ_1 and ϕ_2.

Let $A(\Omega)$ be the sheaf of vector fields orthogonal to the Pfaffian system Ω. The sheaf $A(\Omega)$ is a Lie algebra iff Ω is integrable. In this case there exists a local basis (v_1, v_2) of $A(\Omega)$ such that $[v_1, v_2] = 0$.

Proposition 3.2 Assume (M, g) self-dual. With appropriate coordinates $(\lambda, x_1, x_2, s, t)$ of P, there exists a commuting basis (v_1, v_2) of $A(\Omega)$, in the following form

$$\begin{aligned} v_1 &= \frac{\partial}{\partial s} - \lambda\frac{\partial}{\partial x_1} - \left(\frac{\partial R}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial S}{\partial x_1}\frac{\partial}{\partial x_2} + \frac{\partial T}{\partial x_1}\frac{\partial}{\partial \lambda}\right), \\ v_2 &= \frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial x_2} - \left(\frac{\partial R}{\partial x_2}\frac{\partial}{\partial x_1} + \frac{\partial S}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial T}{\partial x_2}\frac{\partial}{\partial \lambda}\right), \end{aligned} \tag{3.3}$$

where the functions R, S and T do not depend on λ.

Proof. First we notice the following lemma.

Lemma 3.3 ([3]) Let (M, g) be a self-dual Riemannian four-manifold. Then there exist local coordinates (p_1, p_2, q_1, q_2) of M such that

$$g = \sum_{i,j=1,2} P_{ij}(p, q)dp_i dq_j.$$

It follows from Lemma 3.3 that we can take local frames $\{e_j\}_{j=1,2,3,4}$ as follows:

$$\begin{aligned} e_1 &= -dp_1, \ \ e_2 = -(P_{11}dq_1 + P_{12}dq_2), \\ e_4 &= \quad dp_2, \ \ e_3 = \quad P_{21}dq_1 + P_{22}dq_2. \end{aligned}$$

By Theorem 3.1 the ideal $\mathcal{I}$ generated by $(\theta, \sigma_1, \sigma_2)$ is closed under the exterior derivative d. Thus we have

$$\begin{aligned} d\sigma_1 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= 0, \\ d\sigma_2 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 &= 0. \end{aligned} \tag{3.4}$$

By direct calculations we have

$$\begin{aligned} &d\sigma_1 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 \\ &\quad = ((\frac{\partial P_{12}}{\partial q_1} - \frac{\partial P_{11}}{\partial q_2})\mu^2 + K\mu + L) d\mu \wedge dp_1 \wedge dp_2 \wedge dq_1 \wedge dq_2, \\ &d\sigma_2 \wedge \theta \wedge \sigma_1 \wedge \sigma_2 \\ &\quad = ((\frac{\partial P_{22}}{\partial q_1} - \frac{\partial P_{21}}{\partial q_2})\mu^2 + M\mu + N) d\mu \wedge dp_1 \wedge dp_2 \wedge dq_1 \wedge dq_2, \end{aligned}$$

for functions K, L, M and N independent of μ. Thus we have

$$\frac{\partial P_{12}}{\partial q_1} - \frac{\partial P_{11}}{\partial q_2} = 0, \quad \frac{\partial P_{22}}{\partial p_1} - \frac{\partial P_{21}}{\partial p_2} = 0.$$

Hence we can define new coordinates (x_1, x_2, s, t, μ) by the following equations:

$$\begin{gathered} \frac{\partial x_1}{\partial q_i} = -P_{1i}, \quad \frac{\partial x_2}{\partial q_i} = P_{2i}, \ (i = 1, 2) \\ s = p_2, \quad t = -p_1, \end{gathered}$$

The differential forms θ, σ_1 and σ_2 are written in these coordinates as follows:

$$\begin{aligned} \theta &= d\mu + \mu^2(E_1 e_2 + E_2 e_3) + \mu(F_1 e_2 + F_2 e_3) + \sum_j J_j e_j, \\ \sigma_1 &= \mu ds + (dx_1 + A_1 ds + A_2 dt), \\ \sigma_2 &= \mu dt + (dx_2 + B_1 ds + B_2 dt), \end{aligned}$$

For functions $A_j, B_j, E_j, F_j\,(j = 1,2)$, and $J_j\,(j = 1,2,3,4)$ independent of μ. It is easily verified that the following vectors v_1, v_2 belong to $A(\Omega)$:

$$
\begin{aligned}
v_1 &= -\mu\frac{\partial}{\partial x_1} + \frac{\partial}{\partial s} - A_1\frac{\partial}{\partial x_1} - B_1\frac{\partial}{\partial x_2} + (E_1\mu^3 + F_1\mu^2 + J_2\mu - J_4)\frac{\partial}{\partial \mu},\\
v_2 &= -\mu\frac{\partial}{\partial x_2} + \frac{\partial}{\partial t} - A_2\frac{\partial}{\partial x_1} - B_2\frac{\partial}{\partial x_2} + (E_2\mu^3 + F_2\mu^2 + J_3\mu - J_1)\frac{\partial}{\partial \mu}.
\end{aligned}
$$

We set the vector field

$$
\begin{aligned}
l_1 &= \frac{\partial}{\partial s} - A_1\frac{\partial}{\partial x_1} - B_1\frac{\partial}{\partial x_2},\\
l_2 &= \frac{\partial}{\partial t} - A_2\frac{\partial}{\partial x_1} - B_2\frac{\partial}{\partial x_2}.
\end{aligned}
$$

The commutator $[v_1, v_2]$ is written in the following form

$$
\begin{aligned}
[v_1, v_2] =& (E_2\mu^3 + F_2\mu^2)\frac{\partial}{\partial x_1} - (E_1\mu^3 + F_1\mu^2)\frac{\partial}{\partial x_2}\\
&+\{(E_2F_1 - E_1F_2)\mu^4 + (2E_2J_1 - 2E_1J_2 + l_1(E_2) - l_2(E_1))\mu^3\\
&+(F_2J_1 - F_1J_2 + \frac{\partial J_2}{\partial x_2} - \frac{\partial J_3}{\partial x_1} + l_1(F_2) - l_2(F_1))\mu^2\}\frac{\partial}{\partial \mu} + \mu u_1 + u_0,
\end{aligned} \tag{3.5}
$$

where the coefficients of vectors u_1 and u_0 are independent of μ.

By the integrability condition, $[v_1, v_2]$ is a linear combination of v_1 and v_2 and since $[v_1, v_2]$ does not contain $\frac{\partial}{\partial s}$ nor $\frac{\partial}{\partial t}$, v_1 and v_2 commute with each other. It follows from (3.5) that

$$
E_j = 0,\quad F_j = 0\ \ (j = 1,2),\quad \frac{\partial J_2}{\partial x_2} = \frac{\partial J_3}{\partial x_1}.
$$

Thus there exists a function $f = f(x_1, x_2, s, t)$ such that

$$
\frac{\partial f}{\partial x_1} = J_2,\ \ \frac{\partial f}{\partial x_2} = J_3.
$$

We can take new coordinates $(\lambda = \mu + f, x_1, x_2, s, t)$. With these coordinates we have

$$v_1 = \frac{\partial}{\partial s} - \lambda\frac{\partial}{\partial x_1} - (A_1\frac{\partial}{\partial x_1} + B_1\frac{\partial}{\partial x_2} + C_1\frac{\partial}{\partial \lambda}),$$
$$v_2 = \frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial x_2} - (A_2\frac{\partial}{\partial x_1} + B_2\frac{\partial}{\partial x_2} + C_2\frac{\partial}{\partial \lambda}),$$

where $C_1 = \frac{\partial f}{\partial s} - (J_4 + J_2A_1 + J_3B_1)$ and $C_2 = \frac{\partial f}{\partial t} - (J_1 + J_2A_2 + J_3B_2)$. Again taking the coefficients of μ in $[v_1, v_2]$, we obtain that

$$\frac{\partial A_1}{\partial x_2} = \frac{\partial A_2}{\partial x_1}, \ \frac{\partial B_1}{\partial x_2} = \frac{\partial B_1}{\partial x_1}, \ \frac{\partial C_1}{\partial x_2} = \frac{\partial C_1}{\partial x_1}.$$

Thus there exist functions R, S and T which are independent of λ such that

$$\frac{\partial R}{\partial x_j} = A_j, \ \frac{\partial S}{\partial x_j} = B_j, \ \frac{\partial T}{\partial x_j} = C_j \quad \text{for}\, j = 1, 2.$$

This completes the proof of Proposition 3.2. □

Remark. By Theorem 3.1 and Proposition 3.2 the equations of self-dual metrics are equivalent to the compatibility condition $[v_1, v_2] = 0$:

$$\begin{aligned}
&\frac{\partial^2 T}{\partial t\partial x_1} - \frac{\partial^2 T}{\partial s\partial x_2} - \frac{\partial R}{\partial x_2}\frac{\partial^2 T}{\partial x_1^2} \\
&\qquad + (\frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2})\frac{\partial^2 T}{\partial x_1\partial x_2} + \frac{\partial S}{\partial x_1}\frac{\partial^2 T}{\partial x_2^2} = 0, \\
&\frac{\partial^2 R}{\partial t\partial x_1} - \frac{\partial^2 R}{\partial s\partial x_2} - \frac{\partial T}{\partial x_2} - \frac{\partial R}{\partial x_2}\frac{\partial^2 R}{\partial x_1^2} \\
&\qquad + (\frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2})\frac{\partial^2 R}{\partial x_1\partial x_2} + \frac{\partial S}{\partial x_1}\frac{\partial^2 R}{\partial x_2^2} = 0, \\
&\frac{\partial^2 S}{\partial t\partial x_1} - \frac{\partial^2 S}{\partial s\partial x_2} + \frac{\partial T}{\partial x_1} - \frac{\partial R}{\partial x_2}\frac{\partial^2 S}{\partial x_1^2} \\
&\qquad + (\frac{\partial R}{\partial x_1} - \frac{\partial S}{\partial x_2})\frac{\partial^2 S}{\partial x_1\partial x_2} + \frac{\partial S}{\partial x_1}\frac{\partial^2 S}{\partial x_2^2} = 0.
\end{aligned} \tag{3.6}$$

In the equation (2.26) we take $W \in \mathcal{W}_m$. In this case w_1 vanishes. Replacing $g_{0,1}, g_{1,1}$ and $g_{2,1}$ with R, S and T, respectively. The equations (2.27) reduces to (3.6). Therefore we have

Theorem 3.4 The Lie algebra V_m acts on the space of self-dual metrics. This action is transitive.

Let $\mathcal{W}_0$ be the space of $W \in \mathcal{W}$ which commute with D_0. In the equation (2.26) we take $W \in \mathcal{W}_0$. In this case G_0 vanishes. The equation (2.27) reduces to the composed system of the self-dual Yang-Mills equations and the self-dual Einstein equations (see Example 2.1 and 2.2). Let V_0 be the Lie subalgebra of V:

$$\begin{aligned} V_0 = \{ \sum_{0 \le j < r} & F_j(x', D_0) D_j + E(x', D_0); \\ & F_k(x', D_0) \in \mathcal{E} (0 \le k < r),\ E(x', D_0) \in \mathcal{E} \}. \end{aligned}$$

Then V_0 acts on $\mathcal{W}_0$. Thus the self-dual Einstein equations are a specialization of our integrable system.

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